

The Moore-Penrose inverse in rings with involution

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Abstract: Let R be a unital ring with involution. In this paper, several new necessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in a ring R are given. In addition, the formulae of the Moore-Penrose inverse of an element in a ring are presented.

Key words: Moore-Penrose inverse, Group inverse, EP element, Normal element.

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1 Introduction

Let R be a $*$ -ring, that is a ring with an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$. We say that $b \in R$ is the Moore-Penrose inverse of $a \in R$, if the following hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab \quad (ba)^* = ba.$$

There is at most one b such that above four equations hold. If such an element b exists, it is denoted by a^\dagger . The set of all Moore-Penrose invertible elements will be denoted by R^\dagger . An element $b \in R$ is an inner inverse of $a \in R$ if $aba = a$ holds. The set of all inner inverses of a will be denoted by $a\{1\}$. An element $a \in R$ is said to be group invertible if there exists $b \in R$ such that the following equations hold:

$$aba = a, \quad bab = b, \quad ab = ba.$$

The element b which satisfies the above equations is called a group inverse of a . If such an element b exists, it is unique and denoted by $a^\#$. The set of all group invertible elements will be denoted by $R^\#$.

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An element $a \in R$ is called an idempotent if $a^2 = a$. a is called a projection if $a^2 = a = a^*$. a is called normal if $aa^* = a^*a$. a is called a Hermite element if $a^* = a$. a is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$. The set of all EP elements will be denoted by R^{EP} . \tilde{a} is called a $\{1, 3\}$ -inverse of a if we have $a\tilde{a}a = a$, $(a\tilde{a})^* = a\tilde{a}$. The set of all $\{1, 3\}$ -invertible elements will be denoted by $R^{\{1,3\}}$. Similarly, an element $\hat{a} \in R$ is called a $\{1, 4\}$ -inverse of a if $\hat{a}a\hat{a} = a$, $(\hat{a}a)^* = \hat{a}a$. The set of all $\{1, 4\}$ -invertible elements will be denoted by $R^{\{1,4\}}$.

We will also use the following notations: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, $^\circ a = \{x \in R \mid xa = 0\}$ and $a^\circ = \{x \in R \mid ax = 0\}$.

In [2, 12, 13, 20], the authors showed that the equivalent conditions such that $a \in R$ to be an EP element are closely related with powers of the group and Moore-Penrose inverse of a . Motivated by the above statements, in this paper, we will show that the existence of the Moore-Penrose inverse of an element in a ring R is closely related with powers of some Hermite elements, idempotents and projections.

Recently, Zhu, Chen and Patr  cio in [21] introduced the concepts of left $*$ -regular and right $*$ -regular. We call an element $a \in R$ is left (right) $*$ -regular if there exists $x \in R$ such that $a = aa^*ax$ ($a = xaa^*a$). They proved that $a \in R^\dagger$ if and only if a is left $*$ -regular if and only if a is right $*$ -regular. With the help of left (right) $*$ -regular, we will give more equivalent conditions of the Moore-Penrose of an element in a ring.

In [4], Hartwig proved that for an element $a \in R$, a is $\{1, 3\}$ -invertible with $\{1, 3\}$ -inverse x if and only if $x^*a^*a = a$ and it also proved that a is $\{1, 4\}$ -invertible with $\{1, 4\}$ -inverse y if and only if $aa^*y^* = a$. In [14], Penrose proved the following result in the complex matrix case, yet it is true for an element in a ring with involution, $a \in R^\dagger$ if and only if $a \in Ra^*a \cap aa^*R$. In this case, $a^\dagger = y^*ax^*$, where $a = aa^*y = xa^*a$.

It is well-known that an important feature of the Moore-Penrose inverse is that it can be used to represent projections. Let $a \in R^\dagger$, then if we let $p = aa^\dagger$ and $q = a^\dagger a$, we have p and q are projections. In [3], Han and Chen proved that $a \in R^{\{1,3\}}$ if and only if there exists unique projection $p \in R$ such that $aR = pR$. And, it is also proved that $a \in R^{\{1,4\}}$ if and only if there exists unique projection $q \in R$ such that $Ra = Rq$. We will show that the existence of the Moore-Penrose inverse is closely related with some Hermite elements

and projections.

In [7, Theorem 2.4], Koliha proved that $a \in \mathcal{A}^\dagger$ if and only if a is well-supported, where \mathcal{A} is a C^* -algebra. In [8, Theorem 1], Koliha, Djordjević and Cvetković proved that $a \in R^\dagger$ if and only if a is left $*$ -cancellable and well-supported. Where an element $a \in R$ is called well-supported if there exists projection $p \in R$ such that $ap = a$ and $a^*a + 1 - p \in R^{-1}$. In Theorem 3.7, we will show that the condition that a is left $*$ -cancellable in [8, Theorem 1] can be dropped. Moreover, we prove that $a \in R^\dagger$ if and only if there exists $e^2 = e \in R$ such that $ea = 0$ and $aa^* + e$ is left invertible. And, it is also proved that $a \in R^\dagger$ if and only if there exists $b \in R$ such that $ba = 0$ and $aa^* + b$ is left invertible.

In [4], Hartwig proved that $a \in R^{\{1,3\}}$ if and only if $R = aR \oplus (a^*)^\circ$. And, it is also proved that $a \in R^{\{1,4\}}$ if and only if $R = Ra \oplus^\circ (a^*)$. Hence $a \in R^\dagger$ if and only if $R = aR \oplus (a^*)^\circ = Ra \oplus^\circ (a^*)$. We will show that $a \in R^\dagger$ if and only if $R = a^\circ \oplus (a^*a)^n R$. It is also show that $a \in R^\dagger$ if and only if $R = a^\circ + (a^*a)^n R$, for all choices $n \in \mathbf{N}^+$, where \mathbf{N}^+ stands for the set of all positive integers.

2 Preliminaries

In this section, several auxiliary lemmas are presented.

Lemma 2.1. [4] *Let $a \in R$. Then we have the following results:*

- (1) *a is $\{1,3\}$ -invertible with $\{1,3\}$ -inverse x if and only if $x^*a^*a = a$;*
- (2) *a is $\{1,4\}$ -invertible with $\{1,4\}$ -inverse y if and only if $aa^*y^* = a$.*

The following two Lemmas can be found in [14] in the complex matrix case, yet it is true for an element in a ring with involution.

Lemma 2.2. *Let $a \in R$. Then $a \in R^\dagger$ if and only if there exist $x, y \in R$ such that $x^*a^*a = a$ and $aa^*y^* = a$. In this case, $a^\dagger = yax$.*

Lemma 2.3. *Let $a \in R^\dagger$. Then:*

- (1) *$aa^*, a^*a \in R^{EP}$ and $(aa^*)^\dagger = (a^*)^\dagger a^\dagger$ and $(a^*a)^\dagger = a^\dagger (a^*)^\dagger$;*
- (2) *If a is normal, then $a \in R^{EP}$ and $(a^k)^\dagger = (a^\dagger)^k$ for $k \in \mathbf{N}^+$.*

We will give a generalization of Lemma 2.3(1) in the following lemma.

Lemma 2.4. *Let $a \in R^\dagger$ and $n, m \in \mathbf{N}^+$. Then $(aa^*)^n, (a^*a)^m \in R^{EP}$.*

Proof. Suppose $a \in R^\dagger$, by Lemma 2.3 and $(aa^*)^* = aa^*$, we have $((aa^*)^n)^\dagger = ((aa^*)^\dagger)^n$, $((a^*a)^n)^\dagger = ((a^*a)^\dagger)^n$, $(aa^*)^\dagger = (a^*)^\dagger a^\dagger$, $(a^*a)^\dagger = a^\dagger (a^*)^\dagger$, $aa^*(aa^*)^\dagger = (aa^*)^\dagger aa^*$ and $a^*a(a^*a)^\dagger = (a^*a)^\dagger a^*a$. Thus we have

$$(i) \quad (aa^*)^n((aa^*)^n)^\dagger(aa^*)^n = (aa^*)^n((aa^*)^\dagger)^n(aa^*)^n = (aa^*(a^*)^\dagger a^\dagger aa^*)^n = (aa^*)^n;$$

$$(ii) \quad ((aa^*)^n)^\dagger(aa^*)^n((aa^*)^n)^\dagger = ((aa^*)^\dagger)^n(aa^*)^n((aa^*)^\dagger)^n = ((aa^*)^n)^\dagger;$$

$$(iii) \quad [(aa^*)^n((aa^*)^n)^\dagger]^* = [(aa^*)^n((aa^*)^\dagger)^n]^* = [(aa^*(a^*)^\dagger)^n]^* = (aa^*)^n((aa^*)^n)^\dagger;$$

$$(iv) \quad [((aa^*)^n)^\dagger(aa^*)^n]^* = [((aa^*)^\dagger)^n(aa^*)^n]^* = [((aa^*)^\dagger aa^*)^n]^* = ((aa^*)^n)^\dagger(aa^*)^n;$$

$$(v) \quad (aa^*)^n((aa^*)^n)^\dagger = (aa^*)^n((aa^*)^\dagger)^n = (aa^*(a^*)^\dagger)^n = ((aa^*)^\dagger aa^*)^n = ((aa^*)^n)^\dagger(aa^*)^n.$$

By the definition of the EP element, we have $(aa^*)^n \in R^{EP}$. Similarly, $(a^*a)^m \in R^{EP}$. \square

Definition 2.5. *An element $a \in R$ is $*$ -cancellable if $a^*ax = 0$ implies $ax = 0$ and $yaa^* = 0$ implies $ya = 0$.*

The equivalence of conditions (1), (3) and (5) in the following lemma was also proved by Puystjens and Robinson [17, Lemma 3] in categories with involution.

Lemma 2.6. [9] *Let $a \in R$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $a^* \in R^\dagger$;
- (3) a is $*$ -cancellable and aa^* and a^*a are regular;
- (4) a is $*$ -cancellable and a^*aa^* is regular;
- (5) $a \in Ra^*a \cap aa^*R$.

Lemma 2.7. *Let $a \in R^\dagger$ and $n, m \in \mathbf{N}^+$. Then*

- (1) $(aa^*)^n((aa^*)^n)^\dagger a = a$;
- (2) $a((a^*a)^m)^\dagger(a^*a)^m = a$.

Proof. (1) If $n = 1$ and $aa^*(aa^*)^\dagger aa^* = aa^*$, by a is $*$ -cancellable, we have $aa^*(aa^*)^\dagger a = a$. Suppose if $n = k$, we have $(aa^*)^k((aa^*)^k)^\dagger a = a$. By Lemma 2.3, we have

$$\begin{aligned}
& (aa^*)^{k+1}[(aa^*)^{k+1}]^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^{k+1}a = aa^*(aa^*)^k[(aa^*)^\dagger]^k(aa^*)^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^\dagger a a^\dagger) = aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^\dagger)^* a^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^\dagger a a^\dagger)^* a^\dagger a = aa^*(aa^*)^k[(aa^*)^\dagger]^k a a^\dagger (a^\dagger)^* a^\dagger a \\
&= aa^* a a^\dagger (a^\dagger)^* a^\dagger a = aa^*(a a^\dagger)^*(a^\dagger)^*(a^\dagger a)^* \\
&= a(aa^\dagger a)^*(a^\dagger a a^\dagger)^* = aa^*(a^\dagger)^* \\
&= a(a^\dagger a)^* = aa^\dagger a = a.
\end{aligned}$$

Thus by mathematical induction, we have $(aa^*)^n((aa^*)^n)^\dagger a = a$.

(2) It is similar to (1). □

Lemma 2.8. [21] *Let $a \in R$. The following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $a \in aa^*aR$;
- (3) $a \in Raa^*a$.

*In this case, $a^\dagger = (ax)^*axa^* = a^*ya(ya)^*$, where $a = aa^*ax = yaa^*a$.*

Lemma 2.9. [16] *Let $a \in R$. If $aR = a^*R$, then the following are equivalent:*

- (1) $a \in R^{EP}$;
- (2) $a \in R^\dagger$;
- (3) $a \in R^\#$.

3 Main results

In this section, several necessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in a ring R are given.

Theorem 3.1. *Let $a \in R$ and $m, n \in \mathbf{N}^+$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $a \in R(a^*a)^m \cap (aa^*)^n R$;

- (3) $a \in a(a^*a)^n R$;
- (4) $a \in R(aa^*)^n a$;
- (5) $(aa^*)^n \in R^\dagger$ and $(aa^*)^n[(aa^*)^n]^\dagger a = a$;
- (6) $(a^*a)^n \in R^\dagger$ and $a[(a^*a)^n]^\dagger(a^*a)^n = a$;
- (7) a is $*$ -cancellable and $(aa^*)^m$ and $(a^*a)^n$ are regular;
- (8) a is $*$ -cancellable and $(a^*a)^n a^*$ is regular;
- (9) a is $*$ -cancellable and $a^*(aa^*)^n$ is regular;
- (10) a is $*$ -cancellable and $(aa^*)^n \in R^\#$;
- (11) a is $*$ -cancellable and $(a^*a)^n \in R^\#$;
- (12) a is $*$ -cancellable and $(aa^*)^n \in R^\dagger$;
- (13) a is $*$ -cancellable and $(a^*a)^n \in R^\dagger$.

In this case,

$$a^\dagger = y_1^*(aa^*)^{m+n-2}ax_1^* = x_2^*(a^*a)^{2n-1}x_2a^* = a^*y_2(aa^*)^{2n-1}y_2^*, \text{ where } a = x_1(a^*a)^m, a = (aa^*)^ny_1, a = a(a^*a)^nx_2, a = y_2(aa^*)^na.$$

Proof. (1) \Rightarrow (2) By Lemma 2.7 we can get $(aa^*)^n((aa^*)^n)^\dagger a = a$ and $a((a^*a)^m)^\dagger(a^*a)^m = a$. Thus we have $a \in R(a^*a)^m \cap (aa^*)^n R$.

(2) \Rightarrow (1) Suppose $a \in R(a^*a)^m \cap (aa^*)^n R$, then for some $x_1, y_1 \in R$, we have

$$a = x_1(a^*a)^m \quad \text{and} \quad a = (aa^*)^ny_1. \quad (3.1)$$

By (3.1) and Lemma 2.1, we have

$$[x_1(a^*a)^{m-1}]^* \in a\{1, 3\} \quad \text{and} \quad [(aa^*)^{n-1}y_1]^* \in a\{1, 4\}. \quad (3.2)$$

Thus by (3.2) and Lemma 2.2, we have $a \in R^\dagger$ and

$$\begin{aligned} a^\dagger &= a^{(1,4)}aa^{(1,3)} = [(aa^*)^{n-1}y_1]^*a[x_1(a^*a)^{m-1}]^* \\ &= y_1^*(aa^*)^{n-1}a(a^*a)^{m-1}x_1^* = y_1^*(aa^*)^{m+n-2}ax_1^*. \end{aligned}$$

(1) \Rightarrow (3) By Lemma 2.3, we have $a^*a = a^*aa^\dagger(a^\dagger)^*a^*a$ and $a^\dagger(a^\dagger)^*a^*a = a^*aa^\dagger(a^\dagger)^*$.

Thus

$$\begin{aligned}
a &= aa^\dagger a = (aa^\dagger)^* a = (a^\dagger)^* a^* a = (a^\dagger)^* a^* aa^\dagger (a^\dagger)^* a^* a = (a^\dagger)^* (a^* a)^2 a^\dagger (a^\dagger)^* \\
&= ((a^\dagger)^* a^* a) a^* aa^\dagger (a^\dagger)^* = (aa^\dagger a) a^* aa^\dagger (a^\dagger)^* \\
&= aa^* aa^\dagger (a^\dagger)^* = a(a^* aa^\dagger (a^\dagger)^* a^* a) a^\dagger (a^\dagger)^* = a(a^* a)^2 (a^\dagger (a^\dagger)^*)^2 \\
&= \dots \\
&= a(a^* a)^n (a^\dagger (a^\dagger)^*)^n.
\end{aligned}$$

Hence $a \in a(a^* a)^n R$.

(3) \Rightarrow (1) Suppose $a \in a(a^* a)^n R$, then for some $x_2 \in R$ we have $a \in a(a^* a)^n x_2 = aa^* a(a^* a)^{n-1} x_2 \in aa^* aR$. Thus by Lemma 2.8, we have $a \in R^\dagger$ and

$$a^\dagger = [a(a^* a)^{n-1} x_2]^* a(a^* a)^{n-1} x_2 a^* = x_2^* (a^* a)^{n-1} a^* a(a^* a)^{n-1} x_2 a^* = x_2^* (a^* a)^{2n-1} x_2 a^*.$$

(1) \Leftrightarrow (4) It is similar to (1) \Leftrightarrow (3) and suppose $a = y_2(aa^*)^n a$ for some $y_2 \in R$, by Lemma 2.8, we have $a^\dagger = a^* y_2 (aa^*)^{n-1} a [y_2 (aa^*)^{n-1} a]^* = a^* y_2 (aa^*)^{n-1} aa^* (aa^*)^{n-1} y_2^* = a^* y_2 (aa^*)^{2n-1} y_2^*$.

(1) \Rightarrow (5) It is easy to see that by Lemma 2.4 and Lemma 2.7.

(1) \Rightarrow (6) It is similar to (1) \Rightarrow (5).

(5) \Rightarrow (4) Suppose $(aa^*)^n \in R^\dagger$ and $(aa^*)^n ((aa^*)^n)^\dagger a = a$. Let $b = (aa^*)^n [(aa^*)^n]^\dagger$, then $b^* = b$ and $ba = a$. Thus

$$a = ba = b^* a = [(aa^*)^n ((aa^*)^n)^\dagger]^* a = ((aa^*)^n)^\dagger (aa^*)^n a \in R(aa^*)^n a,$$

which imply the condition (4) is satisfied.

(6) \Rightarrow (3) It is similar to (5) \Rightarrow (4).

(1) \Rightarrow (7) It is easy to see that by Lemma 2.4.

(7) \Rightarrow (1) Suppose a is $*$ -cancellable and $(aa^*)^m$ and $(a^* a)^n$ are regular. Then a^* is $*$ -cancellable and $(aa^*)^m ((aa^*)^m)^- (aa^*)^m = (aa^*)^m$, thus $(aa^*)^m ((aa^*)^m)^- (aa^*)^{m-1} a = (aa^*)^{m-1} a$. If $m-1 = 0$, then $(aa^*)^m ((aa^*)^m)^- a = a$, that is $a = aa^* (aa^*)^{m-1} ((aa^*)^m)^- a$, thus by Lemma 2.1, we have $a \in R^{\{1,4\}}$. If $m-1 > 0$, then by a^* is $*$ -cancellable, we have $(aa^*)^m ((aa^*)^m)^- (aa^*)^{m-2} = (aa^*)^{m-2} a$. If $m-2 = 0$, then $(aa^*)^m ((aa^*)^m)^- a = a$, that is $a = aa^* (aa^*)^{m-1} ((aa^*)^m)^- a$, thus by Lemma 2.1, we have $a \in R^{\{1,4\}}$. If $m-2 > 0$,

repeat above steps, we always have $a \in R^{\{1,4\}}$. Similarly, by $(a^*a)^n$ is regular, we always have $a \in R^{\{1,3\}}$. Therefore, by Lemma 2.2, we have $a \in R^\dagger$.

(1) \Rightarrow (8) By Lemma 2.4, we have $(a^*a)^n \in R^{EP}$ and $((a^*a)^n)^\dagger = (a^\dagger(a^*)^\dagger)^n$. Let $c = (a^\dagger)^*((a^*a)^\dagger)^n$, then

$$\begin{aligned}
(a^*a)^n a^* c (a^*a)^n a^* &= (a^*a)^n a^* (a^\dagger)^*((a^*a)^\dagger)^n (a^*a)^n a^* \\
&= (a^*a)^n [a^* (a^\dagger)^* (a^*a)^\dagger] ((a^*a)^\dagger)^{n-1} (a^*a)^n a^* \\
&= (a^*a)^n [a^* (a^\dagger)^* a^\dagger (a^*)^\dagger] ((a^*a)^\dagger)^{n-1} (a^*a)^n a^* \\
&= (a^*a)^n [a^\dagger a a^\dagger (a^*)^\dagger] ((a^*a)^\dagger)^{n-1} (a^*a)^n a^* \\
&= (a^*a)^n (a^*a)^\dagger ((a^*a)^\dagger)^{n-1} (a^*a)^n a^* \\
&= (a^*a)^n ((a^*a)^\dagger)^n (a^*a)^n a^* \\
&= (a^*a)^n ((a^*a)^n)^\dagger (a^*a)^n a^* \\
&= (a^*a)^n a^*.
\end{aligned}$$

Thus $(a^*a)^n a^*$ is regular.

(8) \Rightarrow (7) Suppose a is $*$ -cancellable and $(a^*a)^n a^*$ is regular. Then

$$(a^*a)^n a^* ((a^*a)^n a^*)^- (a^*a)^n a^* = (a^*a)^n a^*,$$

thus $(a^*a)^n a^* ((a^*a)^n a^*)^- (a^*a)^n = (a^*a)^n$ and $(aa^*)^n ((a^*a)^n a^*)^- a^* (aa^*)^n = (aa^*)^n$ by a is $*$ -cancellable, that is $(a^*a)^n$ and $(aa^*)^n$ are regular, therefore the condition (7) is satisfied.

(1) \Leftrightarrow (9) It is similar to (1) \Leftrightarrow (8).

(1) \Rightarrow (10)-(13) It is easy to see that by Lemma 2.4.

The equivalence between (10)-(13) can be seen by Lemma 2.9.

(12) \Rightarrow (9) Suppose a is $*$ -cancellable and $(aa^*)^n \in R^\#$, then

$$(aa^*)^n = (aa^*)^n [(aa^*)^n]^\# (aa^*)^n. \quad (3.3)$$

Pre-multiplication of (3.3) by a^* now yields

$$\begin{aligned}
a^* (aa^*)^n &= a^* (aa^*)^n [(aa^*)^n]^\# (aa^*)^n \\
&= a^* (aa^*)^n [(aa^*)^n]^\# [(aa^*)^n]^\# (aa^*)^n (aa^*)^n \\
&= a^* (aa^*)^n [(aa^*)^n]^\# [(aa^*)^n]^\# (aa^*)^{n-1} a [a^* (aa^*)^n].
\end{aligned}$$

Thus $a^* (aa^*)^n$ is regular. □

Definition 3.2. [18] Let $a, b \in R$, we say that a is a multiple of b if $a \in Rb \cap bR$.

Definition 3.3. Let $a, b \in R$, we say that a is a left (right) multiple of b if $a \in Rb$ ($a \in bR$).

The existence of the Moore-Penrose inverse of an element in a ring is priori related to a Hermite element. If we take $n = 1$, the condition (2) in the following theorem can be found in [18, Theorem 1] in the category case.

Theorem 3.4. Let $a \in R$ and $n \in \mathbf{N}^+$. Then the following conditions are equivalent:

- (1) $a \in R^\dagger$;
- (2) There exists a projection $p \in R$ such that $pa = a$ and p is a multiple of $(aa^*)^n$;
- (3) There exists a Hermite element $q \in R$ such that $qa = a$ and q is a left multiple of $(aa^*)^n$;
- (4) There exists a Hermite element $r \in R$ such that $ra = a$ and r is a right multiple of $(aa^*)^n$;
- (5) There exists $b \in R$ such that $ba = a$ and b is a left multiple of $(aa^*)^n$.

Proof. (1) \Rightarrow (2) Suppose $a \in R^\dagger$ and let $p = aa^\dagger$, then $p^2 = p = p^*$ and $pa = a$. By Lemma 2.3, we have $aa^*(a^\dagger)^*a^\dagger aa^* = aa^*$, $aa^*(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger aa^*$ and $p = aa^\dagger = (aa^\dagger)^* = (a^\dagger)^*a^* = (a^\dagger)^*(aa^\dagger a)^* = (a^\dagger)^*a^\dagger aa^* = (a^\dagger)^*a^\dagger aa^*(a^\dagger)^*a^\dagger aa^* = [(a^\dagger)^*a^\dagger]^2(aa^*)^2 = \dots = [(a^\dagger)^*a^\dagger]^n(aa^*)^n$. By $p = p^*$, we have $p = p^* = [[(a^\dagger)^*a^\dagger]^n(aa^*)^n]^* = (aa^*)^n[(a^\dagger)^*a^\dagger]^n$. Thus p is a multiple of $(aa^*)^n$.

(2) \Rightarrow (3) It is obvious.

(3) \Rightarrow (4) Let $r = q^*$.

(4) \Rightarrow (5) Suppose $r^* = r$, $ra = a$ and r is a right multiple of $(aa^*)^n$, then $r = (aa^*)^nw$ for some $w \in R$. Let $b = r$, then $ba = a$ and by $r^* = r$, we have $b = r = r^* = ((aa^*)^nw)^* = w^*(aa^*)^n$. That is b is a left multiple of $(aa^*)^n$.

(5) \Rightarrow (1) Since b is a left multiple of $(aa^*)^n$, then $b \in R(aa^*)^n$, post-multiplication of $b \in R(aa^*)^n$ by a now yields $ba \in R(aa^*)^na$. Then by $ba = a$, which gives $a \in R(aa^*)^na$, thus the condition (4) in Theorem 3.1 is satisfied. \square

Similarly, we have the following theorem.

Theorem 3.5. *Let $a \in R$ and $n \in \mathbf{N}^+$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) *There exist a projection $w \in R$ such that $aw = a$ and w is a multiple of $(a^*a)^n$;*
- (3) *There exist a Hermite element $u \in R$ such that $au = a$ and u is a right multiple of $(a^*a)^n$;*
- (4) *There exist a Hermite element $v \in R$ such that $av = a$ and v is a left multiple of $(a^*a)^n$;*
- (5) *There exist $c \in R$ such that $ac = a$ and c is a right multiple of $(a^*a)^n$.*

If we take $n = 1$, the condition (2) in the following theorem can be found in [18, Theorem 1] in the category case.

Theorem 3.6. *Let $a \in R$ and $n \in \mathbf{N}^+$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) *There exists a projection $q \in R$ such that $qa = 0$ and $(aa^*)^n + q$ is invertible;*
- (3) *There exists a projection $q \in R$ such that $qa = 0$ and $(aa^*)^n + q$ is left invertible;*
- (4) *There exists an idempotent $f \in R$ such that $fa = 0$ and $(aa^*)^n + f$ is invertible;*
- (5) *There exists an idempotent $f \in R$ such that $fa = 0$ and $(aa^*)^n + f$ is left invertible;*
- (6) *There exists $c \in R$ such that $ca = 0$ and $(aa^*)^n + c$ is invertible;*
- (7) *There exists $c \in R$ such that $ca = 0$ and $(aa^*)^n + c$ is left invertible.*

In this case,

$$a^\dagger = a^*y_i(aa^*)^{2n-1}y_i^*, i \in \{1, 2, 3\}, \text{ where } 1 = y_1((aa^*)^n + q) = y_2((aa^*)^n + f) = y_3((aa^*)^n + c).$$

Proof. (1) \Rightarrow (2) Suppose $a \in R^\dagger$ and let $q = 1 - aa^\dagger$, then $q^2 = q = q^*$ and $qa = 0$. By Lemma 2.3, we have $aa^*(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger aa^*$. Thus, $((aa^*)^n + q)[((a^\dagger)^*a^\dagger)^n + 1 - aa^\dagger] = 1$ and $[((a^\dagger)^*a^\dagger)^n + 1 - aa^\dagger]((aa^*)^n + q) = 1$. Therefore, $(aa^*)^n + p$ is invertible.

(2) \Rightarrow (3) It is clear.

(3) \Rightarrow (1) Suppose $q^2 = q = q^*$, $pa = 0$ and $(aa^*)^n + q$ is left invertible, then $1 = y_1((aa^*)^n + q)$ for some $y_1 \in R$. By $pa = 0$, we have $a = y_1((aa^*)^n + q)a = y_1(aa^*)^na \in R(aa^*)^na$. That is the condition (4) in Theorem 3.1 is satisfied and $a^\dagger = a^*y_1(aa^*)^{n-1}a[y_1(aa^*)^{n-1}a]^* = a^*y_1(aa^*)^{n-1}aa^*(aa^*)^{n-1}y_1^* = a^*y_1(aa^*)^{2n-1}y_1^*$.

(1) \Rightarrow (4) Let $f = q = 1 - aa^\dagger$, then by (1) \Rightarrow (2), which gives $f^2 = f \in R$, $fa = 0$ and $(aa^*)^n + f$ is invertible.

(4) \Rightarrow (5) It is clear.

(5) \Rightarrow (1) Suppose $f^2 = f \in R$, $fa = 0$ and $(aa^*)^n + f$ is left invertible, then $1 = y_2((aa^*)^n + f)$ for some $y_2 \in R$. By $fa = 0$, we have $a = y_2((aa^*)^n + f)a = y_2(aa^*)^na \in R(aa^*)^na$. That is the condition (4) in Theorem 3.1 is satisfied and $a^\dagger = a^*y_2(aa^*)^{n-1}a[y_2(aa^*)^{n-1}a]^* = a^*y_2(aa^*)^{n-1}aa^*(aa^*)^{n-1}y_2^* = a^*y_2(aa^*)^{2n-1}y_2^*$.

(1) \Rightarrow (6) Let $c = q = 1 - aa^\dagger$, then by (1) \Rightarrow (2), which gives $ca = 0$ and $(aa^*)^n + c$ is invertible. Since $c = q$ and $q^2 = q = q^*$, thus $(aa^*)^n + q$ is invertible implies $(aa^*)^n + c$ is invertible.

(6) \Rightarrow (7) It is clear.

(7) \Rightarrow (1) Suppose $ca = 0$ and $(aa^*)^n + c$ is left invertible, then $1 = y_3((aa^*)^n + c)$ for some $y_3 \in R$. By $ca = 0$, we have $a = y_3((aa^*)^n + c)a = y_3(aa^*)^na \in R(aa^*)^na$. That is the condition (4) in Theorem 3.1 is satisfied and

$$a^\dagger = a^*y_3(aa^*)^{n-1}a[y_3(aa^*)^{n-1}a]^* = a^*y_3(aa^*)^{n-1}aa^*(aa^*)^{n-1}y_3^* = a^*y_3(aa^*)^{2n-1}y_3^*.$$

□

Similarly, we have the following theorem.

Theorem 3.7. *Let $a \in R$ and $n \in \mathbf{N}^+$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) *There exists a projection $p \in R$ such that $ap = 0$ and $(a^*a)^n + p$ is invertible;*
- (3) *There exists a projection $p \in R$ such that $ap = 0$ and $(a^*a)^n + p$ is right invertible;*
- (4) *There exists an idempotent $e \in R$ such that $ae = 0$ and $(a^*a)^n + e$ is invertible;*
- (5) *There exists an idempotent $e \in R$ such that $ae = 0$ and $(a^*a)^n + e$ is right invertible;*
- (6) *There exists $b \in R$ such that $ab = 0$ and $(a^*a)^n + b$ is invertible;*
- (7) *There exists $b \in R$ such that $ab = 0$ and $(a^*a)^n + b$ is right invertible.*

In this case,

$a^\dagger = x_i^*(a^*a)^{2n-1}x_ia^*$, $i \in \{1, 2, 3\}$, where $1 = ((aa^*)^n + p)x_1 = ((aa^*)^n + e)x_2 = ((aa^*)^n + b)x_3$.

If we take $n = 1$ in the equivalent condition (2) in Theorem 3.7, we can get the condition a is left $*$ -cancellable in [8, Theorem 1] can be dropped. An element $a \in R$ is called co-supported if there exists a projection $q \in R$ such that $qa = a$ and $aa^* + 1 - q$ is invertible. In [8], Koliha, Djordjević and Cvetkvić also proved that $a \in R^\dagger$ if and only if a is right $*$ -cancellable and co-supported. If we take $n = 1$ in the equivalent condition (2) in Theorem 3.6, we can get the condition a is right $*$ -cancellable can be dropped. Thus we have the following corollary.

Corollary 3.8. *Let $a \in R$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) a is well-supported;
- (3) a is co-supported.

Lemma 3.9. [6] *Let $a, b \in R$. Then we have:*

- (1) $1 - ab$ is left invertible if and only if $1 - ba$ is left invertible;
- (2) $1 - ab$ is right invertible if and only if $1 - ba$ is right invertible;
- (3) $1 - ab$ is invertible if and only if $1 - ba$ is invertible.

In [15], Patrício proved that if $a \in R$ is regular with $a^- \in a\{1\}$, then $a \in R^\dagger$ if and only if $a^*a + 1 - a^-a$ is invertible if and only if $aa^* + 1 - aa^-$ is invertible. In the following theorem, we show that a^*a and aa^* can be replaced by $(a^*a)^n$ and $(aa^*)^n$, respectively. Moreover, the invertibility can be generalized to the one sided invertibility.

Theorem 3.10. *Let $a \in R$ and $n \in \mathbf{N}^+$. If a is regular and $a^- \in a\{1\}$, then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $v = (a^*a)^n + 1 - a^-a$ is invertible;
- (3) $v = (a^*a)^n + 1 - a^-a$ is right invertible;
- (4) $u = (aa^*)^n + 1 - aa^-$ is invertible;
- (5) $u = (aa^*)^n + 1 - aa^-$ is left invertible.

Proof. (2) \Leftrightarrow (4) By $u = (aa^*)^n + 1 - aa^- = 1 + a[a^*(aa^*)^{n-1} - a^-]$ is invertible and Lemma 3.9, we have u is invertible is equivalent to $1 + [a^*(aa^*)^{n-1} - a^-]a = (a^*a)^n + 1 - a^-a$ is invertible, which is v is invertible.

(1) \Rightarrow (2) If $a \in R^\dagger$, then by Lemma 2.8 we have

$$a = x^*aa^*a \text{ and } a = aa^*ay^* \text{ for some } x, y \in R. \quad (3.4)$$

Taking involution on (3.4), we have $a^* = a^*aa^*x$ and $a^* = ya^*aa^*$. Thus

$$a^* = a^*aa^*x = a^*a(a^*aa^*x)x = (a^*a)^2a^*x^2 = \dots = (a^*a)^na^*x^n. \quad (3.5)$$

$$a^* = ya^*aa^* = y(ya^*aa^*)aa^* = y^2a^*(aa^*)^2 = \dots = y^na^*(aa^*)^n. \quad (3.6)$$

By (3.5) and (3.6), we have

$$\begin{aligned} & [(a^*a)^na^*(a^-)^* + 1 - a^*(a^-)^*](a^*x^n(a^-)^* + 1 - a^*(a^-)^*) \\ &= (a^*a)^na^*(a^-)^*a^*x^n(a^-)^* + (a^*a)^na^*(a^-)^*(1 - a^*(a^-)^*) \\ &+ (1 - a^*(a^-)^*)a^*x^n(a^-)^* + (1 - a^*(a^-)^*)^2 \\ &= (a^*a)^na^*(a^-)^*a^*x^n(a^-)^* + (1 - a^*(a^-)^*)^2 \\ &= (a^*a)^na^*x^n(a^-)^* + 1 - a^*(a^-)^* \\ &= a^*(a^-)^* + 1 - a^*(a^-)^* \\ &= 1. \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} & (y^na^*(a^-)^* + 1 - a^*(a^-)^*)[(a^*a)^na^*(a^-)^* + 1 - a^*(a^-)^*] \\ &= y^na^*(a^-)^*(a^*a)^na^*(a^-)^* + y^na^*(a^-)^*(1 - a^*(a^-)^*) \\ &+ (1 - a^*(a^-)^*)(a^*a)^na^*(a^-)^* + (1 - a^*(a^-)^*)^2 \\ &= y^na^*(a^-)^*(a^*a)^na^*(a^-)^* + (1 - a^*(a^-)^*)^2 \\ &= y^na^*(a^-)^*a^*(aa^*)^n(a^-)^* + 1 - a^*(a^-)^* \\ &= a^*(a^-)^* + 1 - a^*(a^-)^* \\ &= 1. \end{aligned} \quad (3.8)$$

By (3.7) and (3.8) we have $(a^*a)^na^*(a^-)^* + 1 - a^*(a^-)^* = 1 + [(a^*a)^n - 1]a^*(a^-)^*$ is invertible. By Lemma 3.9, we have $1 + a^*(a^-)^*[(a^*a)^n - 1] = (a^*a)^n + 1 - a^*(a^-)^*$ is invertible, which is v^* is invertible, thus v is invertible.

(2) \Rightarrow (3) and (4) \Rightarrow (5) It is clear.

(3) \Rightarrow (1) and (5) \Rightarrow (1) It is easy to see that by Theorem 3.7 and Theorem 3.6, respectively. \square

In [4], Hartwig proved that $a \in R^{\{1,3\}}$ if and only if $R = aR \oplus (a^*)^\circ$. And, it is also proved that $a \in R^{\{1,4\}}$ if and only if $R = Ra \oplus^\circ (a^*)$. Hence $a \in R^\dagger$ if and only if $R = aR \oplus (a^*)^\circ = Ra \oplus^\circ (a^*)$.

Theorem 3.11. *Let $a \in R$ and $n \in \mathbf{N}^+$. Then the following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $R = a^\circ \oplus (a^*a)^n R$;
- (3) $R = a^\circ + (a^*a)^n R$;
- (4) $R = (a^*)^\circ \oplus (aa^*)^n R$;
- (5) $R = (a^*)^\circ + (aa^*)^n R$;
- (6) $R =^\circ a \oplus R(aa^*)^n$;
- (7) $R =^\circ a + R(aa^*)^n$;
- (8) $R =^\circ (a^*) \oplus R(a^*a)^n$;
- (9) $R =^\circ (a^*) + R(a^*a)^n$.

Proof. (1) \Rightarrow (2) Suppose $a \in R^\dagger$, then by Theorem 3.1 we have $a \in a(a^*a)^n R$, that is $a = a(a^*a)^n b$ for some $b \in R$. Thus $a[1 - (a^*a)^n b] = 0$, which is equivalent to $1 - (a^*a)^n b \in a^\circ$.

By $1 = 1 - (a^*a)^n b + (a^*a)^n b \in a^\circ + (a^*a)^n R$, we have $R = a^\circ + (a^*a)^n R$. Let $u \in a^\circ \cap (a^*a)^n R$, then we have $au = 0$ and $u = (a^*a)^n v$, for some $v \in R$. Hence $u = (a^*a)^n v = a^*a(a^*a)^{n-1}v = (a(a^*a)^n b)^* a(a^*a)^{n-1}v = b^*(a^*a)^n a^*a(a^*a)^{n-1}v = b^*(a^*a)^n (a^*a)^n v = b^*(a^*a)^n u = b^*(a^*a)^{n-1} a^*(au) = 0$. Whence $R = a^\circ \oplus (a^*a)^n R$.

(2) \Rightarrow (3) It is clear.

(3) \Rightarrow (1) Suppose $R = a^\circ + (a^*a)^n R$, Pre-multiplication of $R = a^\circ + (a^*a)^n R$ by a now yields $aR = aa^\circ + a(a^*a)^n R$. By $aa^\circ = 0$, we have $a \in a(a^*a)^n R$, that is the condition (3) in Theorem 3.1 is satisfied.

By the equivalence between (1), (2) and (3) and Lemma 2.6, which implies the equivalence between (1), (4) and (5). The equivalence between (1), (6)-(9) is similar to the equivalence between (1), (2)-(5). \square

If we take $n = 1$ in the equivalent conditions (2)-(9) in Theorem 3.11, we have the following corollary.

Corollary 3.12. *Let $a \in R$. The following conditions are equivalent:*

- (1) $a \in R^\dagger$;
- (2) $R = a^\circ \oplus a^*aR$;
- (3) $R = a^\circ + a^*aR$;
- (4) $R = (a^*)^\circ \oplus aa^*R$;
- (5) $R = (a^*)^\circ + aa^*R$;
- (6) $R = {}^\circ a \oplus Raa^*$;
- (7) $R = {}^\circ a + Raa^*$;
- (8) $R = {}^\circ (a^*) \oplus Ra^*a$;
- (9) $R = {}^\circ (a^*) + Ra^*a$.

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